Kinematics is the science of motion without regard to the forces giving rise to the motion.

Due to mechanical considerations manipulators are generally constructed with joints exhibiting only one degree of freedom each. Typically, these are rotational or translational.
3. Kinematics

3.1 Forward Kinematics

- To solve the forward kinematics problem we would like to attach a virtual coordinate system to each and every link of the manipulator.

- For reference purposes we may also place frames on the robot's stationary base and out at the gripper (or tool-tip).

- Recalling that an arbitrarily located coordinate frame requires 6-parameters to uniquely describe its pose, it seems counter-intuitive that we will now proceed to use only 4-parameters (the justification for this will follow).

- There exist many approaches to this problem, however, herein we will employ the so-called modified Denavit-Hartenburg approach.

**NOTE:** Be careful as some other textbooks use the standard Denavit-Hartenburg notation!!!
3. Kinematics

3.1 Forward Kinematics

The following figure illustrates the four D-H parameters:

- Link length
- Link offset
- Joint angle
- Link twist
- Link twist
- Axis $i-1$
3. Kinematics

3.1 Forward Kinematics

- **link twist:**
  \[ \alpha_{i-1} \]
  Consider a plane whose normal is the vector \( \vec{a}_{i-1} \) (from \( i - 1 \) to \( i \)), project both axis \( i - 1 \) and axis \( i \) onto this plane, then the angle from axis \( i - 1 \) to axis \( i \) is \( \alpha_{i-1} \) (in the right hand sense). If both axes intersect, then the sign of \( \alpha_{i-1} \) is undefined. (Intuitively, translate axis \( i \) to axis \( i - 1 \) along \( \vec{a}_{i-1} \), then \( \alpha_{i-1} \) is the angle between them).

- **link length:**
  \[ a_{i-1} \]
  The length along the line which is mutually perpendicular to axis \( i - 1 \) and axis \( i \). Assuming that they are not parallel, then the points of intersection on both axes are unique. (Intuitively, think of expanding a cylinder centered around axis \( i - 1 \), draw the radius to the point when it just touches axis \( i \)).

- **link offset:**
  If the link is prismatic then \( d_i \) is the joint variable
  \[ d_i \]
  Consider the point on axis \( i \) where \( \vec{a}_{i-1} \) meets it and also the point where \( \vec{a}_i \) intersects axis \( i \). The signed distance along axis \( i \) from \( \vec{a}_{i-1} \) to \( \vec{a}_i \) is known as the link offset.

- **joint angle:**
  If the link is revolute then \( \theta_i \) is the joint variable
  \[ \theta_i \]
  This is the angle formed between an extension of \( \vec{a}_{i-1} \) and the vector \( \vec{a}_i \) about axis \( i \). (Right hand rule, thumb in the direction of \( \vec{d}_i \), rotate from \( \vec{a}_{i-1} \) to \( \vec{a}_i \)).
3. Kinematics

3.1 Forward Kinematics

- These four parameters \((a_{i-1}, \alpha_{i-1}, d_i, \text{ and } \theta_i)\) constitute the four (modified) Denavit-Hartenberg parameters.

- Having discussed the D-H parameters it is still intuitive to think in terms of transformations between coordinate frames. Thus, we need to select a convention to affix coordinate frames to links.

- Consider the axis \(i-1\) to axis \(i\).
  
  i. Identify the joint axes and draw \(\infty\) lines along them.
  
  ii. Assign the link \((i-1^{th})\) coordinate frame origin at the point along the axis \(i-1\) where the mutual perpendicular (between axis \(i-1\) and axis \(i\)) meets it.

  iii. Assign the \(\hat{z}_{i-1}\) direction along the \(i-1^{th}\) axis.

  iv. Assign the \(\hat{x}_{i-1}\) axis along the mutual perpendicular, from \(\hat{z}_{i-1}\) to \(\hat{z}_i\).

  v. Include the \(\hat{y}_{i-1}\) axis to complete the coordinate system.

  vi. Add a base frame \(\{0\}\) and a tool frame (if desired).
As a simplifying convention for the first (and last) links, we may wish to assign frame \{0\} to match frame \{1\} when the first joint variable is zero (i.e., when $\theta_1 = 0$ or $d_1 = 0$).
We can now construct the homogeneous transformation matrix relating coordinate frames \{i−1\} and \{i\}. To describe \{i\} as seen from \{i−1\}:

1) Rotate about \(\hat{x}_{i−1}\) by \(\alpha_{i−1}\).
2) Then, translate about \(\hat{x}_{i−1}\) by \(a_{i−1}\).
3) Next, rotate about the new z-axis (axis \(i\)) by \(\theta_i\).
4) Finally, translate along the new z-axis (axis \(i\)) by \(d_i\).

Note that these are all relative type transformations!!
3. Kinematics

3.1 Forward Kinematics

Hence,

\[ i^{-1}T = R(\hat{x},a_{i-1}) D(\hat{x},a_{i-1}) R(\hat{z},\theta_i) D(\hat{z},d_i) \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & C_{a_{i-1}} & -S_{a_{i-1}} & 0 \\
0 & S_{a_{i-1}} & C_{a_{i-1}} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & a_{i-1} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
C_{\theta_i} & -S_{\theta_i} & 0 & 0 \\
S_{\theta_i} & C_{\theta_i} & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
C_{\theta_i} & -S_{\theta_i} & 0 & a_{i-1} \\
S_{\theta_i} & C_{\theta_i} & -S_{a_{i-1}} & -S_{a_{i-1}}d_i \\
S_{\theta_i}S_{a_{i-1}} & C_{\theta_i}C_{a_{i-1}} & -S_{a_{i-1}} & -S_{a_{i-1}}d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.1)
3. Kinematics

3.1 Forward Kinematics

Thus, equation (3.1) represents a description of the link coordinate frame \( \{i\} \) in terms of (or as seen from) the link coordinate frame \( \{i - 1\} \), using the 4 D-H parameters.

**Question:**
- How is it that we need only 4 parameters to describe the pose of a frame?

**Answer:**
- By cleverly choosing \( \{i\} \) wrt \( \{i - 1\} \), i.e., we restrict:
  - i) the \( \hat{x}_{i-1} \) axis to intersect \( \hat{z}_i \), and
  - ii) the \( \hat{x}_{i-1} \) axis to be perpendicular to \( \hat{z}_i \).
- These two restrictions effectively constrain the dimensionality of the relationship between the two coordinate frames in question.
- If these two conditions are not upheld then the 4 D-H parameters will **NOT** be sufficient !!!
Remark:

- Notice that steps 1) and 2) can be interchanged (they commute).

\[ R_{(\hat{x}_i, a_{i-1})} D_{(\hat{x}, a_{i-1})} = D_{(\hat{x}_i, a_{i-1})} R_{(\hat{x}, a_{i-1})} \]

- Similarly, steps 3) and 4) can be interchanged (they commute).

\[ R_{(\hat{\hat{z}}, \theta_i)} D_{(\hat{\hat{z}}, d_i)} = D_{(\hat{\hat{z}}, d_i)} R_{(\hat{\hat{z}}, \theta_i)} \]

- In general, rotations and translations about the same axis will commute!!
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Example:

- An example of a 3-degree of freedom (rotary-rotary-prismatic) RRP robot.

  i. Identify the joint axes and draw \( \infty \) lines along them.
ii. Assign the link \((i - 1)^{\text{th}}\) coordinate frame origin at the point along the axis \(i - 1\) where the mutual perpendicular (between axis \(i - 1\) and axis \(i\)) meets it.

**Note:** Since there is no mutual perpendicular between axis 3 and axis 4 (non-existent), the origin of frame \(\{3\}\) can be placed anywhere on axis 3.
iii. Assign the $\hat{Z}_{i-1}$ direction along the $i - 1^{th}$ axis.

**Note:** The direction of $z$-axis is not unique. The negative of what we have chosen would work just fine!!
iv. Assign the $\hat{x}_{i-1}$ axis along the mutual perpendicular, from $\hat{Z}_{i-1}$ to $\hat{Z}_i$.

- For $i = 2$, $\hat{x}_1$ axis from $\hat{Z}_1$ to $\hat{Z}_2$ (unique).

- For $i = 3$, $\hat{x}_2$ axis from $\hat{Z}_2$ to $\hat{Z}_3$ (not unique since axis 2 and axis 3 intersect).

- For $i = 4$, $\hat{x}_3$ axis from $\hat{Z}_3$ to $\hat{Z}_4$ (not unique since axis 4 does not exist).
v. Include the \( \hat{y}_{i-1} \) axis to complete the coordinate system. 
Recall: Right-hand coordinate system.

vi. Add a base frame \( \{0\} \) and a tool frame (if desired).

**Note:** Must adhere to the two D-H constraints when choosing frames \( \{0\} \) & \( \{4\} \)!
Rules to determine the four D-H parameters:

- **Before starting, be sure that for each frame the x-axis points to and is perpendicular to the next z-axis:**
  - This allows us to use only 4-parameters
  - Otherwise DH constraints will **NOT** be upheld
3. Kinematics

3.1 Forward Kinematics

- **Link #1 (From frame \{0\} to frame \{1\} : i=1)**
  - **Link Twist** ($\alpha_{i-1}$):
    - $\alpha_0 = 0^\circ$
    - Rotate about $\hat{x}_0$ by $\alpha_0$ resulting in the z-axis becoming parallel with the next z-axis.
  - **Link Length** ($a_{i-1}$):
    - $a_0 = 0$
    - Translate along $\hat{x}_0$ by $a_0$.
      This will result in the current z-axis becoming collinear with the next z-axis.
  - **Link Offset** ($d_i$):
    - $d_1$
    - Translate along z-axis by $d_1$.
      This will result in the coordinate frame origins becoming coincident.
  - **Joint Angle** ($\theta_i$):
    - $\theta_1(t)$
    - Rotate about z-axis by $\theta_1$.
      This final step should result in frame \{0\} coinciding with frame \{1\}. 
3. Kinematics

3.1 Forward Kinematics

- **Link #2 (From frame \( \{1\} \) to frame \( \{2\} \) : \( i=2 \))**
  - **Link Twist (\( \alpha_{i-1} \))**:
    - \( \alpha_1 = -90^\circ \)
    - Rotate about \( \hat{x}_1 \) by \( \alpha_1 \) resulting in the z-axis becoming parallel with the next z-axis.
  - **Link Length (\( a_{i-1} \))**:
    - Translate along \( \hat{x}_1 \) by \( a_1 \).
    - This will result in the current z-axis becoming collinear with the next z-axis.
  - **Link Offset (\( d_i \))**:
    - Translate along z-axis by \( d_2 \).
    - This will result in the coordinate frame origins becoming coincident.
  - **Joint Angle (\( \theta_i \))**:
    - \( \theta_2(t) - 90^\circ \)
    - Rotate about z-axis by \( \theta_2 \).
    - This final step should result in frame \( \{1\} \) coinciding with frame \( \{2\} \).
3. Kinematics

3.1 Forward Kinematics

- **Link #3 (From frame \{2\} to frame \{3\} : i=3)**
  - Link Twist ($\alpha_{i-1}$):
    - Rotate about $\hat{x}_2$ by $\alpha_2$ resulting in the $z$-axis becoming parallel with the next $z$-axis.
  - Link Length ($a_{i-1}$):
    - Translate along $\hat{x}_2$ by $a_2$.
      This will result in the current $z$-axis becoming collinear with the next $z$-axis.
  - Link Offset ($d_i$):
    - Translate along $z$-axis by $d_3$.
      This will result in the coordinate frame origins becoming coincident.
  - Joint Angle ($\theta_i$):
    - Rotate about $z$-axis by $\theta_3$.
      This final step should result in frame \{2\} coinciding with frame \{3\}. 
3. Kinematics

3.1 Forward Kinematics

- **Link #4 (From frame \{3\} to frame \{4\} : i=4)**
  - **Link Twist \(\alpha_{i-1}\):**
    - \(\alpha_3 = 0^\circ\)
    - Rotate about \(\hat{x}_3\) by \(\alpha_3\) resulting in the z-axis becoming parallel with the next z-axis.
  - **Link Length \(a_{i-1}\):**
    - \(a_3 = 0\)
    - Translate along \(\hat{x}_3\) by \(a_3\).
    - This will result in the current z-axis becoming collinear with the next z-axis.
  - **Link Offset \(d_i\):**
    - \(d_4\)
    - Translate along z-axis by \(d_4\).
    - This will result in the coordinate frame origins becoming coincident.
  - **Joint Angle \(\theta_i\):**
    - \(\theta_4 = 0\)
    - Rotate about z-axis by \(\theta_4\).
    - This final step should result in frame \{3\} coinciding with frame \{4\}.

**NOTE:** There is NO link 4. Both frames \{3\} & \{4\} are attached to link 3.
### 3. Kinematics

#### 3.1 Forward Kinematics

- **NOTE:** Do this one link at a time (i.e. pair of coord frames)

1. All of the above are relative transformations.
2. Either the third or fourth parameter will be the joint variable.

<table>
<thead>
<tr>
<th>D-H params.</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$d_1$</td>
<td>$\theta_1(t)$</td>
</tr>
<tr>
<td>2</td>
<td>$-90$</td>
<td>$a_1$</td>
<td>$d_2$</td>
<td>$\theta_2(t) - 90$</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>0</td>
<td>$d_3(t) + \bar{d}_3$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$d_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

Next, generate the “T-matrices”

- Courtesy of Mathematica
Let us now use the D-H parameters from the link table to construct the homogeneous matrices which will relate one link coordinate frame to the next.

\[
0^1 T(\alpha_0, a_0, d_1, \theta_1(t)) = 0^1 T(0, 0, d_1, \theta_1(t)) = \begin{bmatrix}
C_1 & -S_1 & 0 & 0 \\
S_1 & C_1 & 0 & 0 \\
0 & 0 & 1 & d_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
1^2 T(\alpha_1, a_1, d_2, \theta_2(t)) = 1^2 T(-90, a_1, d_2, \theta_2(t) - 90) = \begin{bmatrix}
S_2 & C_2 & 0 & a_1 \\
0 & 0 & 1 & d_2 \\
C_2 & -S_2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
2^3 T(\alpha_2, a_2, d_3(t), \theta_3) = 2^3 T(90, 0, d_3(t) + \bar{d}_3, 0) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & -d_3 - \bar{d}_3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
3. Kinematics

3.1 Forward Kinematics

Therefore, we can now compute the coordinates of the gripper (frame \{4\}) as seen from the base or reference frame (frame \{0\}), in terms of the 3 joint variables \((\theta_1, \theta_2, d_3)\).

\[
\begin{align*}
{4^3}T(\alpha_3, a_3, d_4, \theta_4) &= {4^3}T(0, 0, d_4, 0) = \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

\[
0^4T = 0^1T \cdot 1^2T \cdot 2^3T \cdot 3^4T
\]

\[
= \begin{bmatrix} C_1S_2 & -S_1 & -C_1C_2 & C_1\left(a_1 - C_2\left(d_3 + d_3 + d_4\right)\right) - d_2S_1 \\ S_1S_2 & C_1 & -C_2S_1 & S_1\left(a_1 - C_2\left(d_3 + d_3 + d_4\right)\right) + C_1d_2 \\ C_2 & 0 & S_2 & S_2\left(d_3 + d_3 + d_4\right) + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}
\]