To perform useful tasks, we will want to place the robot's tool (or gripper) at a desired position, with an appropriate orientation.

By attaching virtual coordinate systems to objects we can describe the positions and orientations of these objects by referring to the poses of the coordinate frame.

**Left Hand vs. Right Hand Coordinate Frame**

- For the purposes of this discussion, there exist two types of coordinate frames:
  - The right hand coordinate frame will be used herein.
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In 3-D space the position of any point \((p\) say\) can be uniquely described by a triplet \(p = (p_x, p_y, p_z)\).
This can be interpreted as a vector

\[ \vec{p} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \quad (2.1) \]

wherein, \( \hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) and are standard unit basis vectors.

From equation (2.1) it can be seen that

\[ p_x = \langle \vec{p}, \hat{x} \rangle = \vec{p} \cdot \hat{x} \]
\[ p_y = \langle \vec{p}, \hat{y} \rangle = \vec{p} \cdot \hat{y} \]
\[ p_z = \langle \vec{p}, \hat{z} \rangle = \vec{p} \cdot \hat{z} \]

Therefore, the description of a point by a vector \( \vec{p} \) is tied to the coordinate system \( (\hat{x}, \hat{y}, \hat{z}) \) in which the vector is described.
Question:

- How can locations described in one coordinate frame be described as seen from another parallel coordinate frame?

Answer:

- The same point ‘p’ can be described $^1\vec{p}$ by in $\{1\}$ \textit{(i.e., sitting on frame \{1\} looking at p)}, or by $^2\vec{p}$ in $\{2\}$ \textit{(i.e., sitting on frame \{2\} looking at p)}.

- Since the two coordinate systems ($\{1\} & \{2\}$) have the same orientation \textit{(i.e., moving in the $\hat{x}_2$ direction is the same as moving in the $\hat{x}_1$ direction)}.

- Hence,

$$^1\vec{p} = ^1\vec{p}_{\text{org}} + ^2\vec{p} \quad \text{(2.2)}$$
Thus, given a position in \( \{2\} \) it can be transformed into a position relative to \( \{1\} \) using (2.2), or vice-versa.

Other 3-tupple descriptions such as cylindrical, spherical, etc. coordinates exist but are more complex to manipulate, in general.
The reference frame \( \{1\} \) is not orientationally aligned with frame \( \{2\} \).

- It is possible to construct a matrix which conveys the description of the orientation of the object frame \( \{2\} \) relative to the reference frame \( \{1\} \) (or vice-versa).

Consider the matrix formed by (sitting on frame \( \{1\} \) looking at frame \( \{2\} \))

\[
\begin{bmatrix}
\hat{x}_2 & \hat{y}_2 & \hat{z}_2 \\
\end{bmatrix}
\]

(2.3)
2. Spatial Relationships

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- The vector $\hat{x}_2$ described in coordinate system $\{1\}$ is

$$\hat{1}_2 = \begin{bmatrix} \langle \bar{x}_2, \bar{x}_1 \rangle \\ \langle \bar{y}_2, \bar{y}_1 \rangle \\ \langle \bar{z}_2, \bar{z}_1 \rangle \end{bmatrix}$$

- Similarly,

$$\hat{1}_2 = \begin{bmatrix} \langle \bar{y}_2, \bar{x}_1 \rangle \\ \langle \bar{y}_2, \bar{y}_1 \rangle \\ \langle \bar{y}_2, \bar{z}_1 \rangle \end{bmatrix}$$

- and

$$\hat{1}_2 = \begin{bmatrix} \langle \bar{z}_2, \bar{x}_1 \rangle \\ \langle \bar{z}_2, \bar{y}_1 \rangle \\ \langle \bar{z}_2, \bar{z}_1 \rangle \end{bmatrix}$$
By reversing the frame of reference (from \( \{1\} \rightarrow \{2\} \)) in order to construct \( 2_1^1 R \), it can be seen that (sitting on frame \( \{2\} \) looking at frame \( \{1\} \))

\[
2_1^1 R = \left[ 1_2^1 R \right]^T = \left[ 1_2^1 R \right]^{-1}
\]  

(2.4)

(\text{note that these rotational matrices are orthonormal, i.e., } R^{-1} = R^T \text{ and } |R| = \pm 1)

- The matrix \( 1_2^1 R \) can be thought to describe the orientation of the object frame \( \{2\} \) relative to the reference (or with respect to) frame \( \{1\} \).
An alternative approach to arriving at equation (2.4) is:

- The point ‘\( p \)’ on the object can be described in the coordinate system \( \{1\} \) as
  \[
  \mathbf{p}_1 = p_{x_1} \mathbf{x}_1 + p_{y_1} \mathbf{y}_1 + p_{z_1} \mathbf{z}_1 \tag{2.5}
  \]

- or in coordinate system \( \{2\} \) as
  \[
  \mathbf{p}_2 = p_{x_2} \mathbf{x}_2 + p_{y_2} \mathbf{y}_2 + p_{z_2} \mathbf{z}_2 \tag{2.6}
  \]
2. Spatial Relationships

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Since $\vec{p}_1$ and $\vec{p}_2$ are representations of the same point but wrt different coordinate systems, then

$$p_{x_1} = \langle \vec{p}_1, \hat{x}_1 \rangle$$

$$= \langle \vec{p}_2, \hat{x}_1 \rangle$$

$$= \langle (p_{x_2} \hat{x}_2 + p_{y_2} \hat{y}_2 + p_{z_2} \hat{z}_2), \hat{x}_1 \rangle$$

$$= p_{x_2} \langle \hat{x}_2, \hat{x}_1 \rangle + p_{y_2} \langle \hat{y}_2, \hat{x}_1 \rangle + p_{z_2} \langle \hat{z}_2, \hat{x}_1 \rangle$$

(2.7)

Similarly, we can show that

$$p_{y_1} = p_{x_2} \langle \hat{x}_2, \hat{y}_1 \rangle + p_{y_2} \langle \hat{y}_2, \hat{y}_1 \rangle + p_{z_2} \langle \hat{z}_2, \hat{y}_1 \rangle$$

(2.8)

$$p_{z_1} = p_{x_2} \langle \hat{x}_2, \hat{z}_1 \rangle + p_{y_2} \langle \hat{y}_2, \hat{z}_1 \rangle + p_{z_2} \langle \hat{z}_2, \hat{z}_1 \rangle$$

(2.9)
2. Spatial Relationships

2.2 3D Orientational Relationships

- Rewriting equations ((2.7) – (2.9)) in matrix form gives:

\[
\begin{bmatrix}
\hat{x}_2 \\
\hat{y}_2 \\
\hat{z}_2
\end{bmatrix}
=\begin{bmatrix}
\begin{pmatrix}
\frac{1}{p_x} \\
p_{z_1}
\end{pmatrix}
&
\begin{pmatrix}
\frac{1}{p_y} \\
p_{z_1}
\end{pmatrix}
&
\begin{pmatrix}
\frac{1}{p_z} \\
p_{z_1}
\end{pmatrix}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{p_x} \\
p_{z_1}
\end{bmatrix}
\]

\[
\Rightarrow \quad \frac{1}{p} = \frac{1}{2} R \frac{2}{p}
\]

- Alternatively, by starting with expressions for \(p_{x_2}, p_{y_2}, p_{z_2}\) and we can show that:

\[
\frac{2}{p} = \frac{2}{1} R \frac{1}{p} = \left[\begin{array}{c}
\frac{1}{2} R
\end{array}\right]^T \frac{1}{p}
\]

- Substituting the expression for \(\frac{2}{p}\) from (2.11) into equation (2.10) suggests that:

\[
\frac{1}{p} = \frac{1}{2} R \left[\begin{array}{c}
\frac{1}{2} R
\end{array}\right]^T \frac{1}{p} \quad \Rightarrow \quad \frac{1}{2} R \left[\begin{array}{c}
\frac{1}{2} R
\end{array}\right] = I
\]

\[
\Rightarrow \quad \left[\begin{array}{c}
\frac{1}{2} R
\end{array}\right]^{-1} = \left[\begin{array}{c}
\frac{1}{2} R
\end{array}\right]^T
\]
Such a matrix is said to be orthogonal. In particular, the column vectors are of unit length and thus rotation matrices are orthonormal.

For a right handed coordinate system it can be shown that

\[ |R| = +1 \]  \hspace{1cm} (2.12)

For our purposes we say that \( R \in SO(3) \), a class of matrices known as the special orthogonal group of order 3.
2.2 3D Orientational Relationships

Yet another way of arriving at the same result:

\[ \hat{\mathbf{p}} = p_{x_1} \hat{x}_1 + p_{y_1} \hat{y}_1 + p_{z_1} \hat{z}_1 \]

\[ = p_{x_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + p_{y_1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + p_{z_1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{x_1} \\ p_{y_1} \\ p_{z_1} \end{bmatrix} \quad (2.12) \]

Similarly, using the frame \{2\} basis vectors described in frame \{1\}

\[ \hat{\mathbf{p}} = p_{x_2} \hat{x}_2 + p_{y_2} \hat{y}_2 + p_{z_2} \hat{z}_2 \]

\[ = \begin{bmatrix} \hat{x}_2 & \hat{y}_2 & \hat{z}_2 \end{bmatrix} \begin{bmatrix} p_{x_2} \\ p_{y_2} \\ p_{z_2} \end{bmatrix} = \begin{bmatrix} \hat{x}_2 & \hat{y}_2 & \hat{z}_2 \end{bmatrix}^2 \hat{\mathbf{p}} \quad (2.13) \]

Hence, (2.13) suggests that

\[ \hat{\mathbf{p}} = \begin{bmatrix} \hat{x}_2 & \hat{y}_2 & \hat{z}_2 \end{bmatrix}^2 \hat{\mathbf{p}} = \hat{\mathbf{p}} R^2 \hat{\mathbf{p}} \]