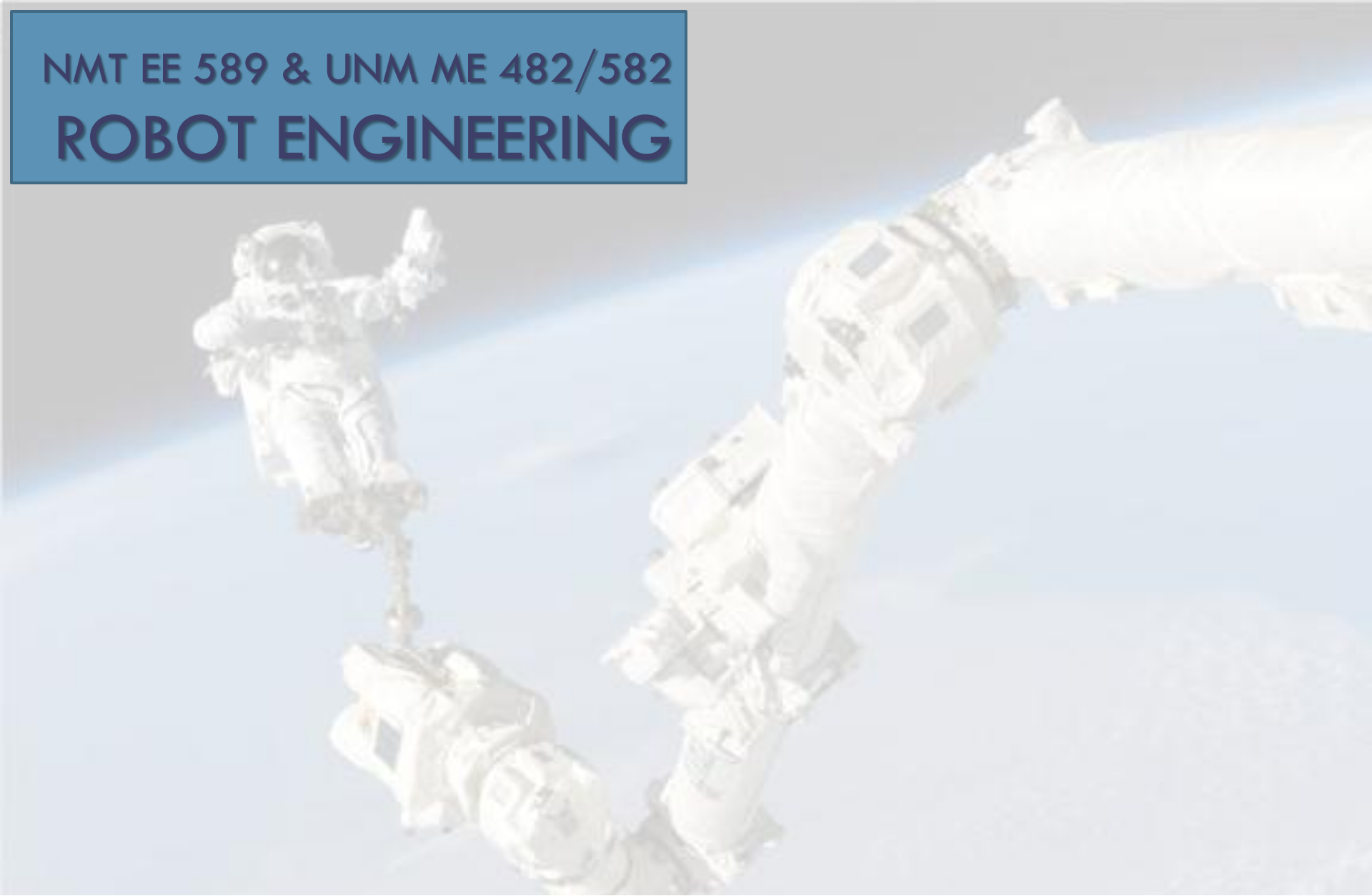
NMT EE 589 & UNM ME 482/582
ROBOT ENGINEERING



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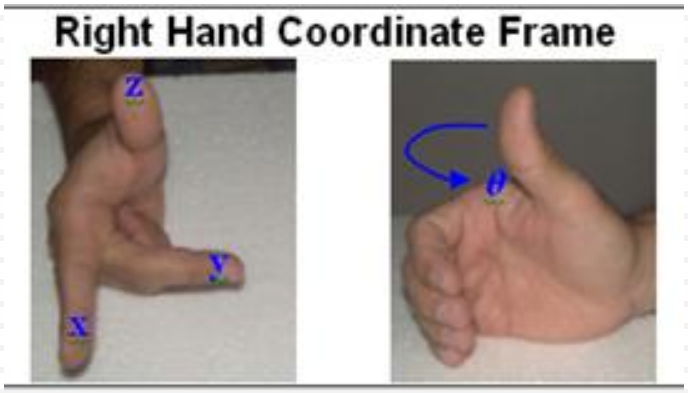
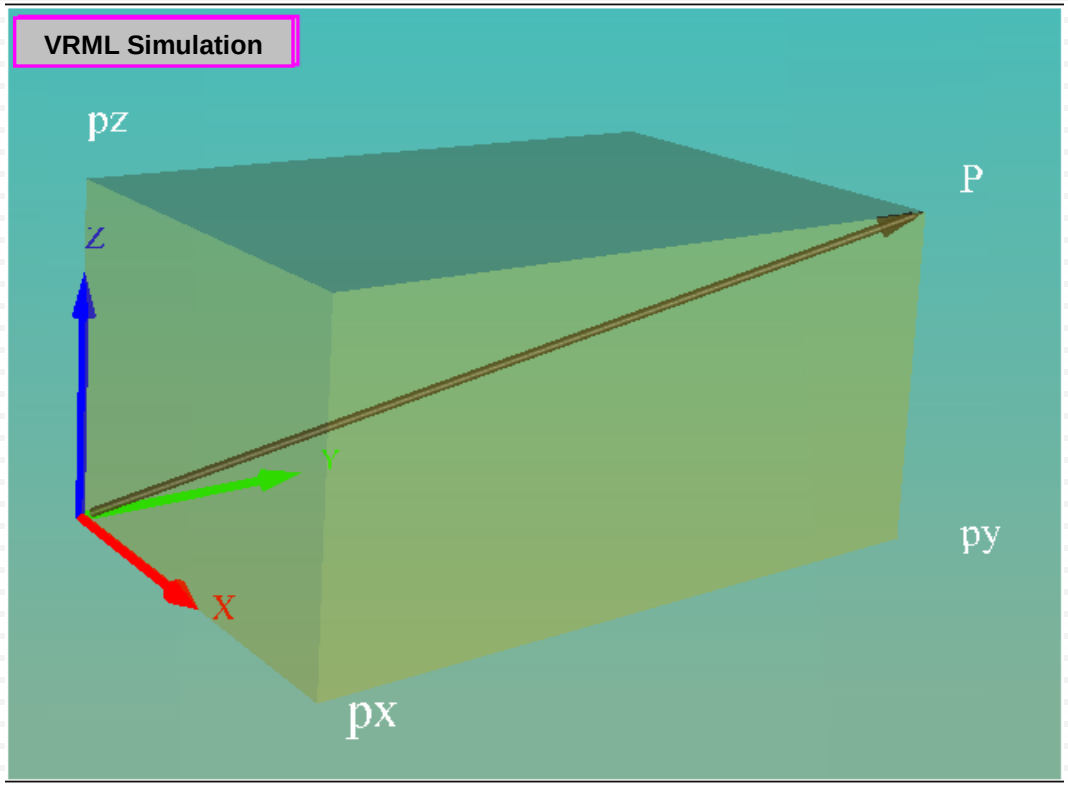
- **To perform useful tasks, we will want to place the robot's tool (or gripper) at a desired position, with an appropriate orientation.**
- **By attaching virtual coordinate systems to objects we can describe the positions and orientations of these objects by referring to the poses of the coordinate frame.**
- **Left Hand vs. Right Hand Coordinate Frame**
  - For the purposes of this discussion, there exist two types of coordinate frames:



  - The right hand coordinate frame will be used herein.

- **In 3-D space the position of any point (*p* say) can be uniquely described by a triplet $p = (p_x, p_y, p_z)$.**



- **This can be interpreted as a vector**
- $$\vec{p} = p_x\hat{\vec{x}} + p_y\hat{\vec{y}} + p_z\hat{\vec{z}} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \textbf{(2.1)}$$

  wherein, $\hat{\vec{x}} \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\vec{y}} \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{\vec{z}} \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **and are standard unit basis vectors.**
- **From equation (2.1) it can be seen that**

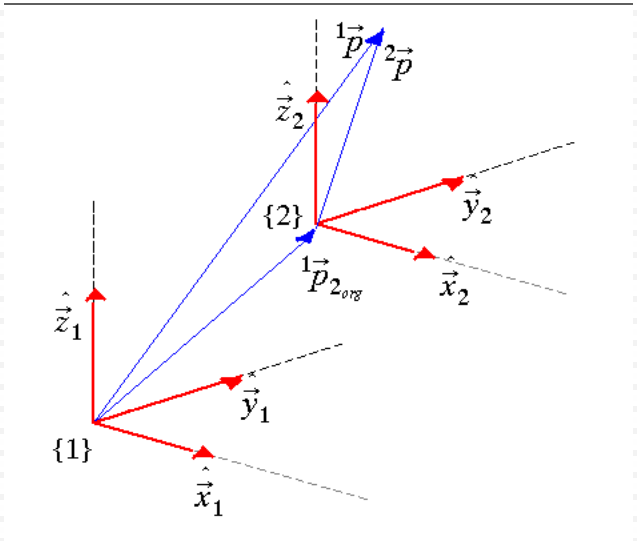
$$p_x = <\vec{p}, \hat{\vec{x}}> = \vec{p} \bullet \hat{\vec{x}}$$

$$p_y = <\vec{p}, \hat{\vec{y}}> = \vec{p} \bullet \hat{\vec{y}}$$

$$p_z = <\vec{p}, \hat{\vec{z}}> = \vec{p} \bullet \hat{\vec{z}}$$

- **Therefore, the description of a point by a vector $\vec{p}$ is tied to the coordinate system $\left(\hat{\vec{x}}, \hat{\vec{y}}, \hat{\vec{z}}\right)$ in which the vector is described.**

- **Question:**
  - How can locations described in one coordinate frame be described as seen from another parallel coordinate frame?
- **Answer:**
  - The same point '*p*' can be described ${}^{1}\vec{p}$ by in {1} (*i.e.,* sitting on frame {1} looking at *p*), or by ${}^{2}\vec{p}$ in {2} (*i.e.*, sitting on frame {2} looking at *p*).
  - Since the two coordinate systems ({1} & {2}) have the same orientation (*i.e.,* moving in the $\hat{\vec{x}}_2$ direction is the same as moving in the $\hat{\vec{x}}_1$ direction).
  - Hence,

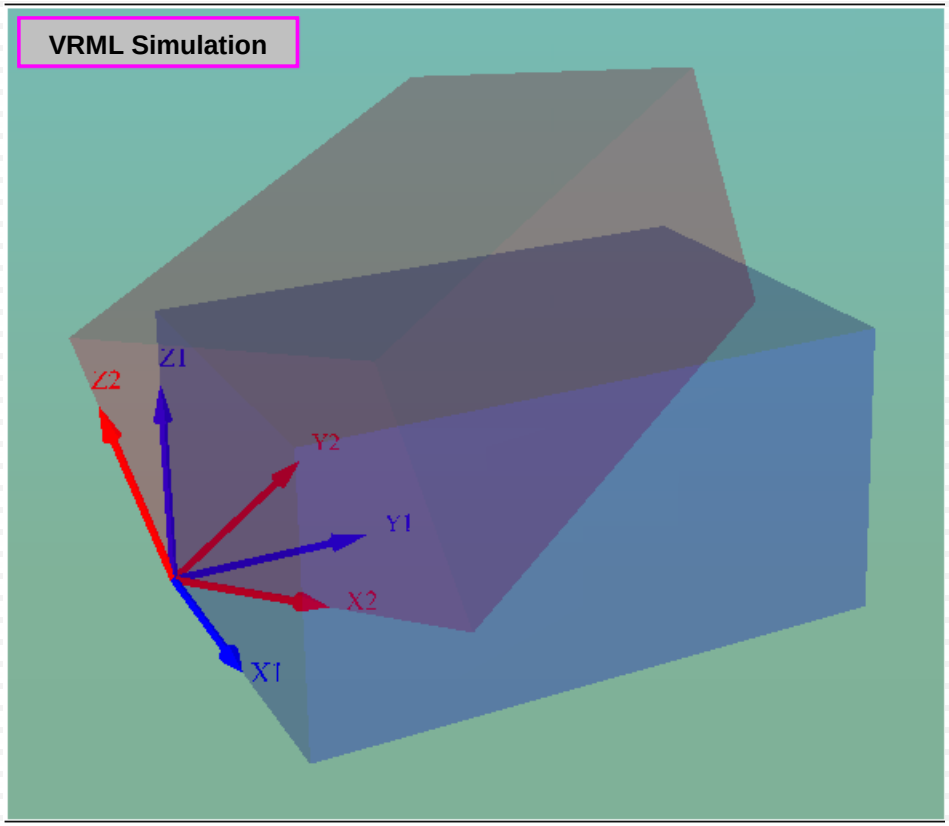


$$ {}^{1}\vec{p} = {}^{1}\vec{p}_{2_{org}} + {}^{2}\vec{p} \qquad \textbf{(2.2)} $$

- Thus, given a position in {2} it can be transformed into a position relative to {1} using (2.2), or vice-versa.
- Other 3-tupple descriptions such as cylindrical, spherical, *etc.* coordinates exist but are more complex to manipulate, in general.

2.2 3D Orientational Relationships

- **The reference frame {1} is not orientationally aligned with frame {2}.**
  - It is possible to construct a matrix which conveys the description of the orientation of the object frame {2} relative to the reference frame {1} (or vice-versa).
- **Consider the matrix formed by (sitting on frame {1} looking at frame {2})**
- $${}^{1}_{2}R \doteq \left[{}^{1}\hat{\vec{x}}_2, {}^{1}\hat{\vec{y}}_2, {}^{1}\hat{\vec{z}}_2 \right] \qquad \textbf{(2.3)}$$



- **The vector $\hat{\vec{x}}_2$ described in coordinate system {1} is**

$$
{}^{1}\hat{\vec{x}}_2 = <\vec{x}_2, \vec{x}_1> \hat{\vec{x}}_1 + <\vec{x}_2, \vec{y}_1> \hat{\vec{y}}_1 + <\vec{x}_2, \vec{z}_1> \hat{\vec{z}}_1
$$

$$
= \begin{bmatrix} <\vec{x}_2, \vec{x}_1> \\ <\vec{x}_2, \vec{y}_1> \\ <\vec{x}_2, \vec{z}_1> \end{bmatrix}
$$

- **Similarly,**

$$
{}^{1}\hat{\vec{y}}_2 = <\vec{y}_2, \vec{x}_1> \hat{\vec{x}}_1 + <\vec{y}_2, \vec{y}_1> \hat{\vec{y}}_1 + <\vec{y}_2, \vec{z}_1> \hat{\vec{z}}_1
$$

$$
= \begin{bmatrix} <\vec{y}_2, \vec{x}_1> \\ <\vec{y}_2, \vec{y}_1> \\ <\vec{y}_2, \vec{z}_1> \end{bmatrix}
$$

- **and**

$$
{}^{1}\hat{\vec{z}}_2 = <\vec{z}_2, \vec{x}_1> \hat{\vec{x}}_1 + <\vec{z}_2, \vec{y}_1> \hat{\vec{y}}_1 + <\vec{z}_2, \vec{z}_1> \hat{\vec{z}}_1
$$

$$
= \begin{bmatrix} <\vec{z}_2, \vec{x}_1> \\ <\vec{z}_2, \vec{y}_1> \\ <\vec{z}_2, \vec{z}_1> \end{bmatrix}
$$

- **By reversing the frame of reference (from {1} → {2}) in order to construct ${}^{2}_{1}R$, it can be seen that (sitting on frame {2} looking at frame {1})**

$$ {}^{2}_{1}R = \left[{}^{1}_{2}R \right]^{T} = \left[{}^{1}_{2}R \right]^{-1} \qquad \textbf{(2.4)} $$

- **(note that these rotational matrices are orthonormal, *i.e.*, $R^{-1} = R^{T}$ and $|R| = \pm 1$)**
  - The matrix ${}^{1}_{2}R$ can be thought to describe the orientation of the object frame {2} relative to the reference (or with respect to) frame {1}.

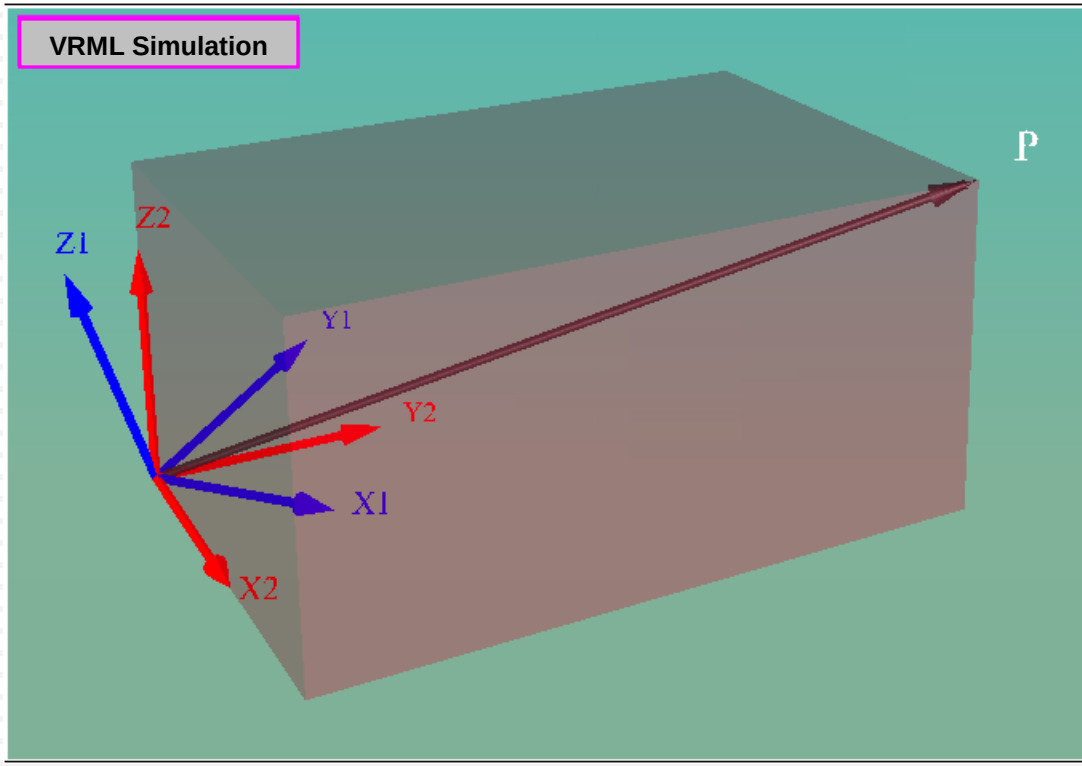
An alternative approach to arriving at equation (2.4) is:

- The point '*p*' on the object can be described in the coordinate system {1} as

$$^{1}\vec{p} = p_{x_1}\hat{\vec{x}}_1 + p_{y_1}\hat{\vec{y}}_1 + p_{z_1}\hat{\vec{z}}_1 \qquad (2.5)$$

- or in coordinate system {2} as

$$^{2}\vec{p} = p_{x_2}\hat{\vec{x}}_2 + p_{y_2}\hat{\vec{y}}_2 + p_{z_2}\hat{\vec{z}}_2 \qquad (2.6)$$



- **Since ${}^{1}\vec{p}$ and ${}^{2}\vec{p}$ are representations of the same point but *wrt* different coordinate systems, then**

$$
\begin{aligned}
p_{x_1} &= <{}^{1}\vec{p}, \hat{\vec{x}}_1> \\
&= <{}^{2}\vec{p}, {}^{2}\hat{\vec{x}}_1> \\
&= <(p_{x_2}\hat{\vec{x}}_2 + p_{y_2}\hat{\vec{y}}_2 + p_{z_2}\hat{\vec{z}}_2), {}^{2}\hat{\vec{x}}_1> \\
&= p_{x_2} <\hat{\vec{x}}_2, {}^{2}\hat{\vec{x}}_1> + p_{y_2} <\hat{\vec{y}}_2, {}^{2}\hat{\vec{x}}_1> + p_{z_2} <\hat{\vec{z}}_2, {}^{2}\hat{\vec{x}}_1>
\end{aligned}
\qquad \textbf{(2.7)}
$$

- **Similarly, we can show that**

$$p_{y_1} = p_{x_2} <\hat{\vec{x}}_2, {}^{2}\hat{\vec{y}}_1> + p_{y_2} <\hat{\vec{y}}_2, {}^{2}\hat{\vec{y}}_1> + p_{z_2} <\hat{\vec{z}}_2, {}^{2}\hat{\vec{y}}_1> \qquad \textbf{(2.8)}$$

$$p_{z_1} = p_{x_2} <\hat{\vec{x}}_2, {}^{2}\hat{\vec{z}}_1> + p_{y_2} <\hat{\vec{y}}_2, {}^{2}\hat{\vec{z}}_1> + p_{z_2} <\hat{\vec{z}}_2, {}^{2}\hat{\vec{z}}_1> \qquad \textbf{(2.9)}$$

- **Rewriting equations ((2.7) – (2.9)) in matrix form gives**

$$\begin{bmatrix} p_{x_1} \\ {}^1\vec{p} \\ p_{z_1} \end{bmatrix} = \begin{bmatrix} < {}^1\hat{\vec{x}}_2, {}^2\hat{\vec{x}}_1 > & < {}^1\hat{\vec{y}}_2, {}^2\hat{\vec{x}}_1 > & < {}^1\hat{\vec{z}}_2, {}^2\hat{\vec{x}}_1 > \\ < {}^1\hat{\vec{x}}_2 & > & < {}^1\hat{\vec{y}}_2 & > & < {}^1\hat{\vec{z}}_2 & > \\ < x_2, z_1 > & < y_2, z_1 > & < z_2, z_1 > \end{bmatrix} \begin{bmatrix} p_{x_2} \\ {}^2\vec{p} \\ p_{z_2} \end{bmatrix}$$

$$\therefore \qquad {}^1\vec{p} = {}^1_2R\,{}^2\vec{p} \qquad \textbf{(2.10)}$$

- **Alternatively, by starting with expressions for p_{x_2}, p_{y_2}, p_{z_2} and we can show that**

$${}^2\vec{p} = {}^2_1R\,{}^1\vec{p} = \left[{}^1_2R\right]^T {}^1\vec{p} \qquad \textbf{(2.11)}$$

- **Substituting the expression for ${}^2\vec{p}$ from (2.11) into equation (2.10) suggests that**

$${}^1\vec{p} = {}^1_2R\left[{}^1_2R\right]^T {}^1\vec{p} \qquad \Rightarrow \qquad {}^1_2R\left[{}^1_2R\right] = I$$

$$\therefore \qquad \left[{}^1_2R\right]^{-1} = \left[{}^1_2R\right]^T$$

- **Such a matrix is said to be orthogonal. In particular, the column vectors are of unit length and thus rotation matrices are orthonormal.**
- **For a right handed coordinate system it can be shown that**

$$|R| = +1 \tag{2.12}$$

  - For our purposes we say that $R \in SO(3)$, a class of matrices known as the special orthogonal group of order 3.

What is the determinant of R for a left-handed coordinate system?

- **Yet another way of arriving at the same result;**

$$ {}^{1}\vec{p} = p_{x_1}\,{}^{1}\hat{\vec{x}}_1 + p_{y_1}\,{}^{1}\hat{\vec{y}}_1 + p_{z_1}\,{}^{1}\hat{\vec{z}}_1 $$

$$ = p_{x_1}\begin{bmatrix}1\\0\\0\end{bmatrix} + p_{y_1}\begin{bmatrix}0\\1\\0\end{bmatrix} + p_{z_1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}p_{x_1}\\p_{y_1}\\p_{z_1}\end{bmatrix} \qquad \textbf{(2.12)} $$

- **Similarly, using the frame {2} basis vectors described in frame {1}**

$$ {}^{1}\vec{p} = p_{x_2}\,{}^{1}\hat{\vec{x}}_2 + p_{y_2}\,{}^{1}\hat{\vec{y}}_2 + p_{z_2}\,{}^{1}\hat{\vec{z}}_2 \qquad \textbf{(2.13)} $$

$$ = \begin{bmatrix} {}^{1}\hat{\vec{x}}_2 & {}^{1}\hat{\vec{y}}_2 & {}^{1}\hat{\vec{z}}_2 \end{bmatrix}\begin{bmatrix}p_{x_2}\\p_{y_2}\\p_{z_2}\end{bmatrix} = \begin{bmatrix} {}^{1}\hat{\vec{x}}_2 & {}^{1}\hat{\vec{y}}_2 & {}^{1}\hat{\vec{z}}_2 \end{bmatrix} {}^{2}\vec{p} $$

- **Hence, (2.13) suggests that**

$$ {}^{1}\vec{p} = \begin{bmatrix} {}^{1}\hat{\vec{x}}_2 & {}^{1}\hat{\vec{y}}_2 & {}^{1}\hat{\vec{z}}_2 \end{bmatrix} {}^{2}\vec{p} = {}^{1}_{2}R\,{}^{2}\vec{p} $$