It is often desirable to minimize the time to traverse a joint trajectory.

- Would like to move at a max constant joint rate.
- To minimize time employ max joint acceleration to get to this constant joint rate.
6. Trajectory Generation

6.3.3 A Linear Segment With Parabolic Blends

- Assume that we start from a position \( \theta(t_0) \doteq \theta_0 \) and come to rest at a final point \( \theta(t_f) \doteq \theta_f \).

- Furthermore, assume that the manipulator was initially at rest \( \dot{\theta}(t_0) = 0 \), and that it will come to rest at the final point \( \dot{\theta}(t_f) = 0 \).
  - Let’s allow \( t_0 = 0 \), for simplicity’s sake!! (can always compensate afterwards)

**Segment 1:**

- Assume a constant acceleration of \( \ddot{\theta}(t) \doteq a \). Using a Taylor series expansion:

  \[
  \theta(t) = \theta_0 + \dot{\theta}(t_0)[t - t_0] + \ddot{\theta}(t_0)\frac{(t - t_0)^2}{2}, \quad t_0 \leq t \leq t_b
  \]

  \[
  = \theta_0 + a \frac{t^2}{2}, \quad t \in [0, t_b)
  \]

- Thus, the position at the end of this segment is

  \[
  \theta(t_b) = \theta_0 + a \frac{t_b^2}{2} \doteq \theta_1
  \]

(6.5)
6. Trajectory Generation

6.3.3 A Linear Segment With Parabolic Blends

- The velocity at any time during this segment is
  \[ \dot{\theta}(t) = at \]

- Thus, the velocity at the end of this blend segment is
  \[ \dot{\theta}(t_b) = a t_b \hat{=} v \quad \text{(say)} \tag{6.6} \]

Segment 2:

- A constant velocity of \( \dot{\theta}(t) \hat{=} v \)
  \[ \theta(t) = \theta_1 + \dot{\theta}(t_b)[t - t_b] \quad , \quad t_b \leq t \leq t_f - t_b \]
  \[ \Rightarrow \quad \theta(t_f - t_b) = \theta_1 + [t_f - 2t_b]v \hat{=} \theta_2 \tag{6.7} \]

- and
  \[ \dot{\theta}(t) = \dot{\theta}(t_b) = v \quad , \quad t_b \leq t \leq t_f - t_b \]
Segment 3:

- A constant acceleration of $\ddot{\theta}(t) \doteq -a$

  $$\theta(t) = \theta_2 + \dot{\theta}(t_f - t_b)[t - [t_f - t_b]] + \ddot{\theta}(t_f - t_b) \frac{(t - [t_f - t_b])^2}{2} \quad , (t_f - t_b) \leq t \leq t_f$$

  $$= \theta_2 + v[t - [t_f - t_b]] - a \frac{(t - [t_f - t_b])^2}{2}$$

- Thus, the position at the end of this segment is

  $$\theta(t_f) = \theta_2 + vt_b - a \frac{t_b^2}{2} \doteq \theta_f$$  \hspace{1cm} (6.8)

- and the velocity at the end of this blend segment is

  $$\dot{\theta}(t) \bigg|_{t=t_f} = v - a[t - [t_f - t_b]] \bigg|_{t=t_f}$$

  $$= v - at_b = 0$$
○ Note that: Given $\theta_0$, $\theta_f$, $t_0$, and $t_f$, if we choose to decrease the magnitude of the acceleration ($|a|$) then segments [1] and [3] become larger and segment [2] becomes smaller.

○ If we choose the magnitude of the acceleration during the blend periods to be too small, then a viable solution may cease to exist!!

**Question:**

○ What is the bound on the value of the acceleration for a viable LSPB?

**Let’s use equations (6.5–6.8) to develop a relationship between $t_b$ and the acceleration $a$.** Substituting for $\theta_2$ in equation (6.8) using equation (6.7) we get,

\[
\theta_f = \theta_1 + v[t_f - 2t_b] + vt_b - a\frac{t_b^2}{2}
\]

\[
\theta_f = \theta_2 + vt_b - a\frac{t_b^2}{2}
\]

\[
\theta_2 = \theta_1 + [t_f - 2t_b]v
\]

**then, substituting for $\theta_1$ using (6.5) we get,**

\[
\theta_f = \theta_0 + a\frac{t_b^2}{2} + v[t_f - 2t_b] + vt_b - a\frac{t_b^2}{2}
\]

\[
= \theta_0 + v[t_f - t_b]
\]

\[
\theta_t = \theta_0 + a\frac{t_b^2}{2}
\]
Recalling the expression for $v$ from equation (6.6) gives

$$\theta_f = \theta_0 + at_b [t_f - t_b] \quad v = at_b$$

$$\Rightarrow at_b^2 - at_f t_b + [\theta_f - \theta_0] = 0$$

Solving for this quadratic in $t_b$ gives

$$t_b = \frac{at_f \pm \sqrt{a^2 t_f^2 - 4a[\theta_f - \theta_0]}}{2a}$$

$$= \frac{t_f}{2} - \frac{\sqrt{a^2 t_f^2 - 4a[\theta_f - \theta_0]}}{2a}$$

- Intuitively, from the diagram if $t_b \geq \frac{t_f}{2}$, then a viable solution ceases to exist!! or $a^2 t_f^2 - 4a[\theta_f - \theta_0] > 0$

$$\therefore a \geq \frac{4[\theta_f - \theta_0]}{t_f^2} \quad (6.9)$$
At the point where the equality holds \( a = \frac{4(\theta_f - \theta_0)}{t_f^2} \), the linear segment is of length zero. This is known as a bang-bang parabolic blend!!
6. Trajectory Generation

6.3.3 A Linear Segment With Parabolic Blends

A Bang-Bang Parabolic Blend

\[ a \geq \frac{4[\theta_f - \theta_0]}{t_f^2} = 2.96 \degree / s^2 \]
Now specify the path in term of the tool pose

Can describe straight line Cartesian trajectories

Both position and orientation

Requires an inverse kinematic computation at each point

Describing the orientation can be challenging

- Euler angles?
- RPY angles?
- Rotation matrix?
- Angle/axis representation?
A Puma 560 example

- Compare a Joint space vs Cartesian space trajectory

MATLAB Example “Traj_gen1.m”
6. Trajectory Generation

6.4 An Example: Joint vs Cartesian

Joint Space Trajectory Generation

Cartesian Space Trajectory Generation