The rotation matrix \((R)\) is a 3×3 matrix requiring 9 parameters to describe an orientation. It can be shown that any orientation can be uniquely described using only 3-parameters.

Some 3-parameters descriptions include:

1. Roll, pitch, yaw angles
2. Euler angles
3. Equivalent angle-axis format

Example:

- Rotate frame \(\{2\}\) by \(\theta\) degrees about the z-axis.
Now, we know that $\frac{1}{2}R \doteq [\hat{1}x_2, \hat{1}y_2, \hat{1}z_2]$. From the diagram above we see that:

$$\hat{1}x_2 = \cos(\theta)\hat{x}_1 + \sin(\theta)\hat{y}_1 + 0\hat{z}_1 \quad \text{(2.13)}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos(\theta) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \sin(\theta) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{(2.14)}$$

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \quad \text{(2.15)}$$

$$\hat{1}y_2 = -\sin(\theta)\hat{x}_1 + \cos(\theta)\hat{y}_1 + 0\hat{z}_1 \quad \text{(2.16)}$$

$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \quad \text{(2.17)}$$
2. Spatial Relationships

2.3 Minimal Descriptions of Orientation

\[
\begin{align*}
\hat{z}_2 &= 0\hat{x}_1 + 0\hat{y}_1 + 1\hat{z}_1 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\end{align*}
\]

\[
1R \doteq \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{(\hat{z}, \theta)}
\]

- Alternatively, we can derive the same result by noting that

\[
< \hat{x}_2, \hat{x}_1 > \doteq \hat{x}_2 \cdot \hat{x}_1 \\
= |\hat{x}_2| \cdot |\hat{x}_1| \cdot \cos(\theta) \\
= \cos(\theta) \\
< \hat{x}_2, \hat{y}_1 > = \cos(90^\circ - \theta) = \sin(\theta) \\
< \hat{x}_2, \hat{z}_1 > = \cos(90^\circ) = 0
\]
2. Spatial Relationships

2.3 Minimal Descriptions of Orientation

\[ \langle \hat{y}_2, \hat{x}_1 \rangle = \cos(90^\circ + \theta) = -\sin(\theta) \]

\[ \langle \hat{y}_2, \hat{y}_1 \rangle = \cos(\theta) \]

\[ \langle \hat{y}_2, \hat{z}_1 \rangle = \cos(90^\circ) = 0 \]

\[ \langle \hat{z}_2, \hat{x}_1 \rangle = \cos(90^\circ) = 0 \]

\[ \langle \hat{z}_2, \hat{y}_1 \rangle = \cos(90^\circ) = 0 \]

\[ \langle \hat{z}_2, \hat{z}_1 \rangle = \cos(0^\circ) = 1 \]

- And recalling that

\[ \mathbf{R} = \begin{bmatrix}
\langle \hat{x}_2, \hat{x}_1 \rangle & \langle \hat{x}_2, \hat{y}_1 \rangle & \langle \hat{x}_2, \hat{z}_1 \rangle \\
\langle \hat{y}_2, \hat{x}_1 \rangle & \langle \hat{y}_2, \hat{y}_1 \rangle & \langle \hat{y}_2, \hat{z}_1 \rangle \\
\langle \hat{z}_2, \hat{x}_1 \rangle & \langle \hat{z}_2, \hat{y}_1 \rangle & \langle \hat{z}_2, \hat{z}_1 \rangle 
\end{bmatrix} \]

\[ \mathbf{R} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix} = \mathbf{R}_{(\hat{z}, \theta)} \quad (2.21) \]
Similarly, it can be shown that (remember: right handed coordinate system)

\[
R_{(\hat{x}, \theta)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\theta) & -\sin(\theta) \\
0 & \sin(\theta) & \cos(\theta)
\end{bmatrix}
\] (2.22)

\[
R_{(\hat{y}, \theta)} = \begin{bmatrix}
\cos(\theta) & 0 & \sin(\theta) \\
0 & 1 & 0 \\
-\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\] (2.23)

A Rotation matrix can be interpreted in 3 distinct ways:

1. IT REPRESENTS A COORDINATE TRANSFORMATION RELATING THE COORDINATES OF A POINT ‘P’ (SAY) IN TWO DIFFERENT FRAMES OF REFERENCE.

2. IT DESCRIBES THE ORIENTATION OF ONE COORDINATE FRAME WRT ANOTHER COORDINATE FRAME.

3. IT IS AN OPERATOR TAKING A VECTOR \( \vec{p} \) AND ROTATING IT INTO A NEW VECTOR \( R\vec{p} \), BOTH IN THE SAME COORDINATE SYSTEM.

Depending on the problem being considered one of the above interpretations may be more intuitively appealing than the others.
Question:

- Does the order of rotations matter (i.e., do they commute)?

Answer:

Consider an object, and a fixed point ‘p’ on the object.

- First, rotate the object about the z-axis of a fixed coordinate frame by \( \theta \) degrees. The new location of the point \( p \) is
  \[
  \vec{p}_1 = R_{(\hat{z}, \theta)} \vec{p}
  \]

- Then, rotate the object about the y-axis of the original fixed coordinate frame by \( \varphi \) degrees, the point \( p \) is now located at
  \[
  \vec{p}_2 = R_{(\hat{y}, \varphi)} \vec{p}_1 = \begin{bmatrix} R_{(\hat{y}, \varphi)} & R_{(\hat{z}, \theta)} \end{bmatrix} \vec{p} = R_{\alpha} \vec{p}
  \]
The composite rotation matrix becomes

\[
R_a = \begin{bmatrix}
  R_{(\hat{y},\varphi)} & R_{(\hat{z},\theta)} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \cos(\varphi) & 0 & \sin(\varphi) \\
  0 & 1 & 0 \\
  -\sin(\varphi) & 0 & \cos(\varphi) \\
\end{bmatrix}
\begin{bmatrix}
  \cos(\theta) & -\sin(\theta) & 0 \\
  \sin(\theta) & \cos(\theta) & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \cos(\varphi)\cos(\theta) & -\cos(\varphi)\sin(\theta) & \sin(\varphi) \\
  \sin(\theta) & \cos(\theta) & 0 \\
  -\sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & \cos(\varphi) \\
\end{bmatrix}
\]

Suppose we had first rotated about the y-axis by \( \varphi \) and then about the z-axis by \( \theta \), then the composite rotation would be

\[
R_b = \begin{bmatrix}
  R_{(\hat{z},\theta)} & R_{(\hat{y},\varphi)} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \cos(\theta)\cos(\varphi) & -\sin(\theta) & \cos(\theta)\sin(\varphi) \\
  \sin(\theta)\cos(\varphi) & \cos(\theta) & \sin(\theta)\sin(\varphi) \\
  -\sin(\varphi) & 0 & \cos(\varphi) \\
\end{bmatrix}
\]

Remark: Therefore, it is clear that rotations do not commute, in general.
The RPY (roll, pitch, yaw) description of orientation is sometimes called X-Y-Z fixed angles, because, they describe consecutive rotations about each axis \( \text{wrt} \) a fixed datum.

- First, roll about the (reference \( \{0\} \)) \( x \)-axis (\( \hat{x}_0 \)) by an angle of \( \gamma \) degrees.
  - This takes us to frame \( \{1\} \), note that \( \hat{x}_0 \) coincides with \( \hat{x}_1 \).
Then, pitch about the (reference \{0\}) y-axis ($\hat{y}_0$) by an angle of $\beta$ degrees. This takes us from frame \{1\} to frame \{2\}. 
2. Spatial Relationships

2.3.1 Fixed Axis: Roll, Pitch, and Yaw Angles

- Finally, yaw about the (reference \{0\}) $z$-axis ($\hat{z}_0$) by an angle of $\alpha$ degrees.

- This takes us from frame \{2\} to frame \{3\}.
Putting them all together, we go from frame {0} to frame {1} to frame {2} to frame {3}.
Thus, the final result becomes

$$R_{r,p,y} = R = R_{(\gamma,\alpha)} R_{(\beta,\gamma)} R_{(\alpha,\gamma)}$$

$$R_{r,p,y} = R = \begin{bmatrix}
    C_\alpha C_\beta & -S_\alpha C_\gamma + C_\alpha S_\beta S_\gamma & S_\alpha S_\gamma + C_\alpha S_\beta C_\gamma \\
    S_\alpha C_\beta & C_\alpha C_\gamma + S_\alpha S_\beta S_\gamma & -C_\alpha S_\gamma + S_\alpha S_\beta C_\gamma \\
    -S_\beta & C_\beta S_\gamma & C_\beta C_\gamma
\end{bmatrix}$$

(2.24)

- Interpretation #3 works best here
- Think of rotating the original x, y, and z original basis vectors

Use the MATLAB **ROBOTICS TOOLBOX** to confirm this result for a few arbitrary RPY angles and generate other fixed axis rotation matrices yourself.

- fixed_axis_example1.m
- fixed_axis_example2.m

Remember to set the MATLAB “Path” to where you have placed the “rvctools” folder!!
Experiment with fixed axis rotation by using the VRML simulation below. Use the x, y, and z sliders to set the roll, pitch, and yaw angles, respectively. You choose the order.
Question:

Given an arbitrary rotation matrix

\[
R = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]

what are the roll, pitch, yaw angles needed to realize this rotation matrix?

Answer:

Comparing equations (2.24) and (2.25), it can be seen that

\[
r_{11} = C_\alpha C_\beta
\]

\[
r_{21} = S_\alpha C_\beta
\]

\[
\Rightarrow \alpha = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) = A \tan 2(r_{21}, r_{11}) \quad \text{(if } C_\beta \neq 0)\]
2. Spatial Relationships

2.3.1 Fixed Axis: Roll, Pitch, and Yaw Angles

- Also, observe that
  \[ r_{32} = C_\beta S_\gamma \]
  \[ r_{33} = C_\beta C_\gamma \]
  \[ \Rightarrow \gamma = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) = A \tan 2(r_{32}, r_{33}) \quad \text{(if } C_\beta \neq 0) \]

- Finally, from noting that
  \[ r_{31} = -S_\beta \]
  \[ r_{11}^2 + r_{21}^2 = C_\beta^2 \]
  \[ \Rightarrow \beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right) = A \tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \]

- **Question?**
  - What happens if \( C_\beta = 0 \) (i.e., \( \beta = \pm 90^\circ \))?